Getting the Rescaled Width Function and the Derived WGIUH

Riccardo RIGON, Andrea COZZINI, Silvano PISONI

Dipartimento Ingegneria Civile e Ambientalee CUDAM, Universita' di Trento, via Mesiano 77,
tel. 0461-882610, fax 0461-882672, e-mail :riccardo.rigon@ing.unitn.it

Abstract
It is shown how rescaled width functions (Rinaldo et al., 1995; D'Odorico et al., 1996) are built by means of Grass and some other custom programs. From them the width function based instantaneous unit hydrograph, WGIUH, is obtained. In the end, it is discussed the possible generalization of the procedure to include spatial soil moisture variability.

Width function and rescaled width function
A fundamental property of drainage networks is that there is a unique one-dimensional path, connecting any point with the outlet (and any other point in the basin) obtained following the steepest descent. The fraction of points at the same distance to the outlet measured along these paths is called width function (Rodriguez-Iturbe et Rinaldo, 1997) and can be espressed as

$$W(x) \equiv \frac{1}{A}\mu\{a : d(a) = x\} \qquad [1]$$

where A is the total area of the basin which normalizes the measure, μ, of the area, *a*, at distance *x* (*d(a)=x*) from the outlet of which a representation is given in Figure 1a. Figure 1b is the correspondent width function.
Width functions depend on the structure of the flow paths which are known to have fractal characteristics (Rodriguez-Iturbe e Rinaldo, 1997) and inherits from them interesting mathematical properties which were widely studied in literature either in theoretical contexts (Rodriguez-Iturbe et Rinaldo, 1997; Gupta et al., 1996; Veneziano et al., 2000; Marani et al., 1992; Marani et al, 1991) or to obtain insight into the flood formation processes (Rinaldo et al, 1991; Rigon et al, 1996; D'Odorico et al, 1998).
Due to the balancing of channel slope, geometry and roughness, the flow velocity tends to remain constant along the network during events of the same return time (e.g. Bathurst, 1993). Thus, the distances can be mapped into time and the distances in Figure 1(a) can be substituted by the isochrones (e.g., Maione et Moisello, 1993), the points the flow of which arrives at the outlet at the same time. The width function in Figure 1(b) is consequently mapped into the width function based instantaneous unit hydrograph:

$$\begin{aligned} &WGIUH(t;c) = W(x(t)) \cdot c \\ &x(t) = c \cdot t \end{aligned} \qquad [2]$$

where the multiplication by the celerity, c, is necessary to maintain the correct normalization, $\int_0^\infty WGIUH(t;c)dt = \int_0^\infty W(x)dx = 1$. The WGIUH is the shape of the flood wave as produced by an impulsive effective rain (Dingman, 1994). The above picture is however physically incorrect. The runoff in hillslope is known in fact to have very slower celerities and higher residence times than the flow in channels (Van der Tak et Bras; 1988; Emmet, 1978): the celerity of the flood wave in channels is usually around 1-5 m/s, while the runoff in hillslope and unchannellized valleys is at least ten times slower.

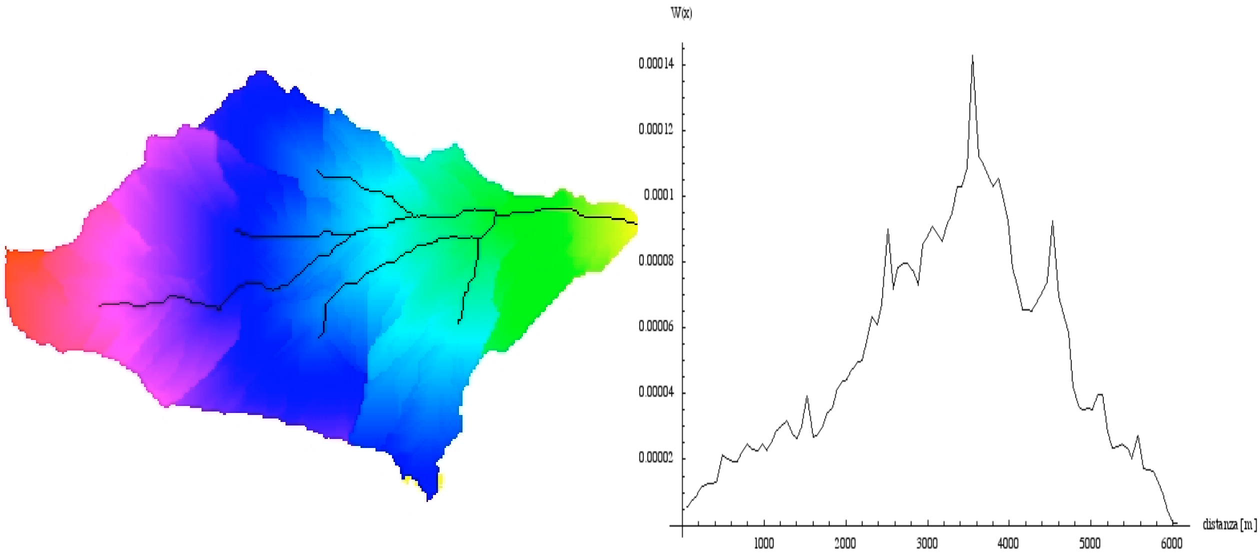

Figure 1– a) Distances from the outlet in the Renanchio (TO) watershed. Going from green to red, the distance increases. It can be observed that spatially close points can have very different distances as shown by the abrupt variation of colors. In black the river network.. Drainage directions in hillslopes (not drawn) are determined according to the steepest descent. b) The resulting W(x) obtained from the distances shown in 1.a.

As shown in Rinaldo et al (1995) e D'Odorico et al. (1998), it is however possible to account for the two celerities and obtain hydrologically sound results introducing a rescaled width function. Let $r=c/c_h$ where c_h is the mean celerity in hillslopes (and, as usual, c is celerity of water in channels), first a rescaled distance, x', is obtained by multiplying by r times the part of distance to outlet covered in hillslopes (red in Figure 2 for a sample point the path of which subsequently continues along the yellow line). Thus:

$$x' = x_c + r \cdot x_h \quad [3]$$

where x_h and x_c are the distance of the given point to the channel and the distance of the point of flow into the channel from the outlet. Then, if we apply (1) to the rescaled distances, it is obtained the *rescaled width function* (Rinaldo et al, 1995), *W'(x')*, which is represented for the Renanchio watershed in Figure 3.

The rescaled width function (Figura 3) and the width function (Figure 1b) have very different skewess: while the original width function was almost symmetric, the rescaled width function is very left skewed and looks much more like the actual flood waves. With the application of (2), but starting from the rescaled width function, we obtain a realistic WGIUH which was used to reproduce real flood events (D'Odorico e al., 1998).

The rescaled width function and discharges

In Figure 4 the rescaled isochrones relative to the rescaled width function in Figure 3 are mapped: the isochrones follow the river network development and are more spaced in hillslopes.

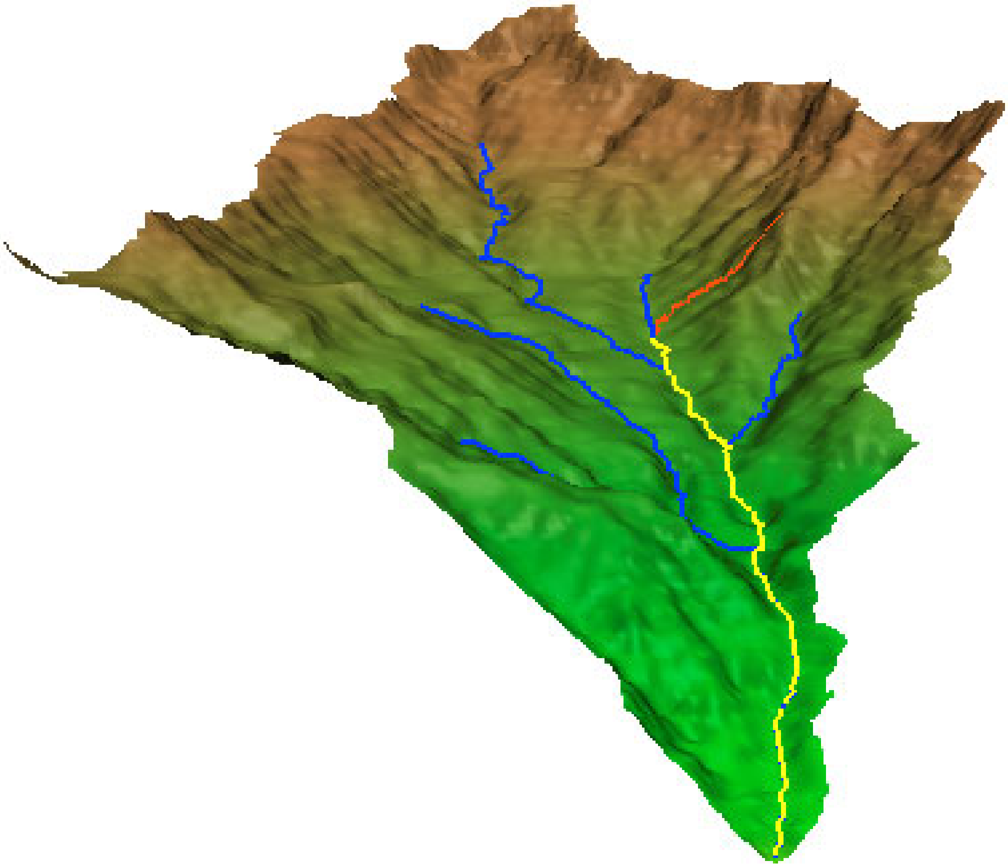

Figure 2– It shows Renanchio basin in 3D with its network. The red line is the path from an arbitrary point to the channel and the yellow line marks the remaining part of the pathway to the outlet. The first, red part of the path, is covered with a slower velocity than the yellow part in the channel.

The discharge at the outlet, according to the theory of the geomorphologic instantaneous unit hydrograph and neglecting the contribution of the subsurface flow, is:

$$Q(t;c,r) = A_T \int_0^t WGIUH(t-\tau;c,r) J_{eff}(\tau) d\tau \qquad [4]$$

where J_{eff} is the effective rainfall, i.e. the rainfall minus the interception loss on vegetation cover (infiltration in soils is usually also subtracted, but if infiltration is assumed approximately in equilibrium with the subsurface flow into channel, it can be neglected). If we assume time invariant WGIUH and the effective rainfall given by:

$$J_{eff}(t) = \begin{cases} J_{eff} & 0 \le t \le t_p \\ 0 & otherwise \end{cases} \qquad [5]$$

the discharge become (Rigon et D'Odorico, 2001):

$$Q(t;c,r) = \begin{cases} J_{eff} A(t;c,r) & 0 \le t \le t_p \\ J_{eff} [A(t;c,r) - A(t-t_p;c,r)] & t > t_p \end{cases} \qquad [6]$$

where $A(t;c,r)$ is the area contributing to discharge at time t.

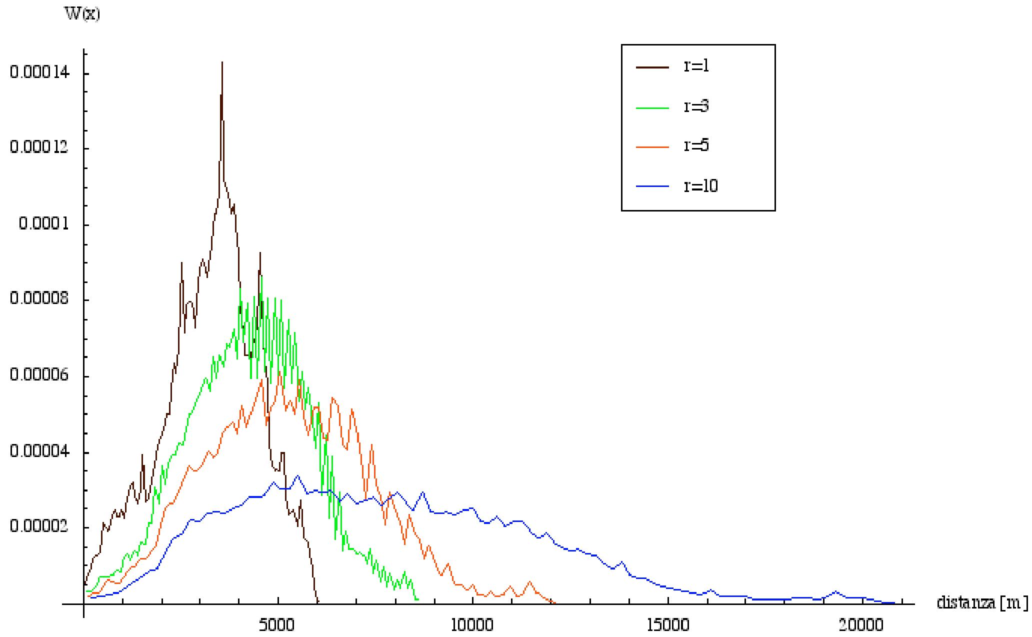

Figure 3 – The rescaled width function for different values of r. The value of r=1 results in the ordinary width function.

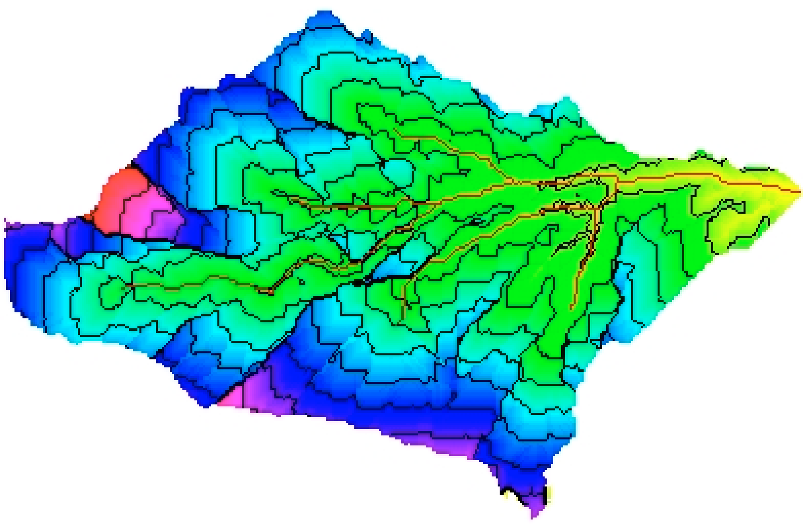

Figure 4 – Renanchio isochrones obtained with r=10 and flood wave celerity equal to 1 m/s. The river network is in red.

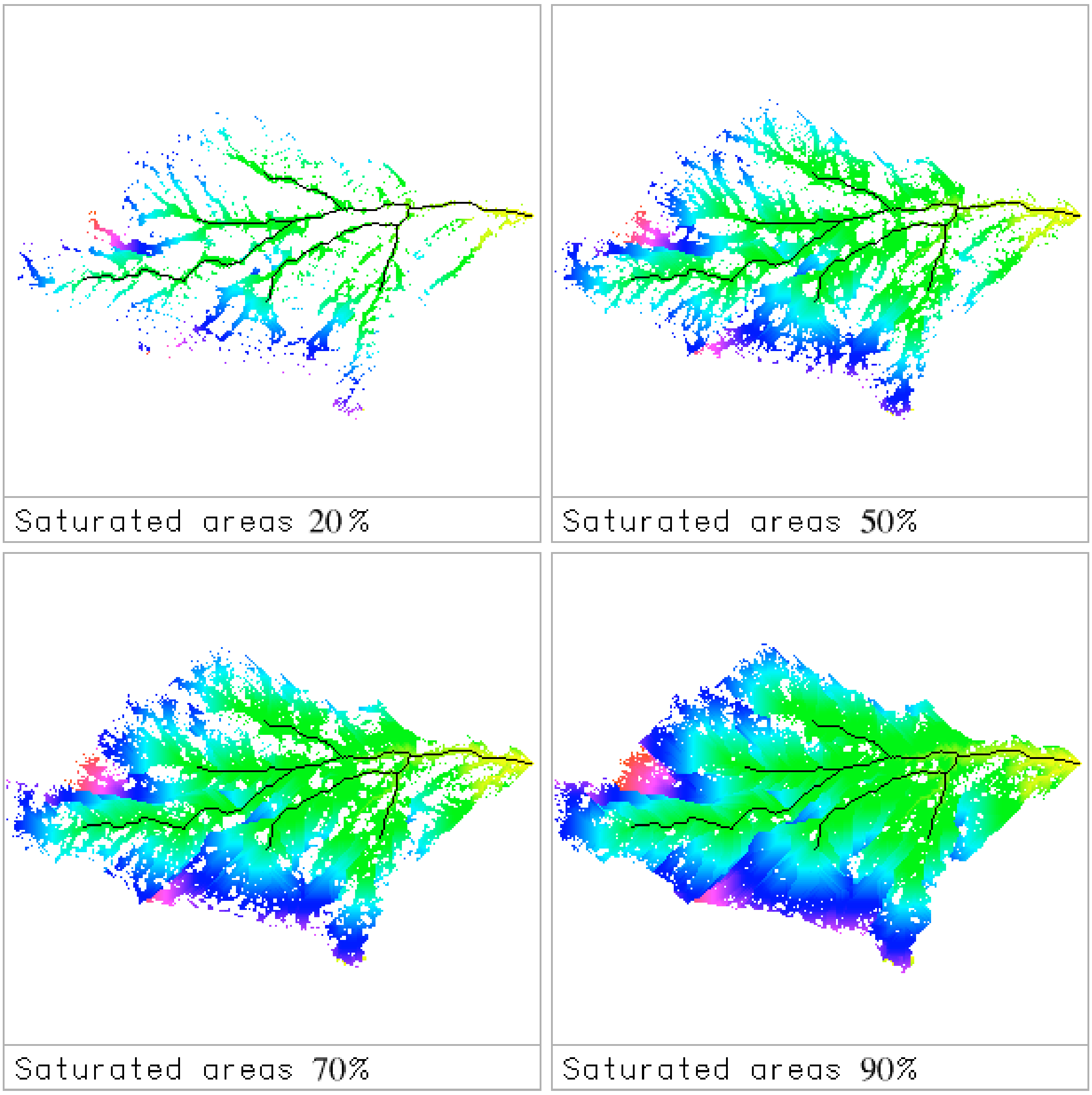

Figure 5 – Saturated areas when 10%, 20%, 50%, 90% of the basin is saturated.

The crux of the matter of making the discharge calculated by means of (4) realistic is the choice of the celerity *c* and the parameter *r*. Such parameters are often determined by calibration against data coming from some events, the effective rainfall and the discharge at the outlet of which are known (Rodriguez-Iturbe e Rinaldo, 1997; D'Odorico e al, 1996). A finer treatment of celerities would make them an increasing function of rainfall volumes: this would make the theory non linear and *geomorphoclimatic* (Rodriguez-Iturbe et al,1982a 1982b).
Therefore introduced rescaled width functions were calculated considering the whole basin area contributing to floods formation. It is known however that this is really the special case which, according to the modern conceptualisation of the phenomena of overland flow, is called Hortonian. In humid areas the Dunnian case is instead common, in which runoff occurs mainly over already saturated areas (Dunne, 1978). The Hortonian case occurs when the rainfall intensity is larger than the soil infiltrability. A simple look at the soil saturated conductivities, as reported by hydrology manuals (e.g. Dingman, 1994), implies that rainfalls usually easily infiltrate and only a few events of very high return time are affected by diffuse Hortonian runoff. As a consequence most of the runoff occurs where infiltrated water accumulates, i.e. in depressions or zones with low or absent

slope. The Dunnian phenomenology is enhanced by the fast decrease of hydraulic conductivity with soil depth which causes the formation of a shallow water table closed to the terrain surface, the top of which can arrive at the surface after a few rainfall events sufficiently close in time. *Return flow* from up-hill can also help the increase of the soil moisture content till saturation.

Thus, an approximate but realistic representation of areas contributing to floods must collect only those parts of the basins which are saturated. According to the TOPMODEL (e.g. Beven et Kirkby, 1979: Beven, 2001; Franchini, 1996), basin points which saturated first are those for which the *topographic index* is higher:

$$I_T = \ln\left(\frac{A}{b\nabla z}\right) - \lambda \qquad [7]$$

where A is the contributing area in each point of the basin, b a unit contour width through which the area drains, ∇z the local slope and λ is the basin averaged $\ln\left(A/b\nabla z\right)$. All these quantities can be derived from DEM standard analyses (they correspond to the Grass commands: r.watershed, r.topidx, r.slope.aspect, r.water.outlet). Figure 5 shows all the points correspondent respectively to the 10,50, 70 and 90 percent of the basin saturation. In fact, starting from the maps of the topographic index, it is possible to get the probability $P[I_T > i]$ of having a point in the basin with topographic index larger than the value i and, viceversa, we can extract those points of the map for which, fixed the quantile q, we have $P[I_T > i(q)]=q$. Assumed Dunnian runoff, we could hence consider the rescaled width function relative only to saturated points. In Figure 6 we have the rescaled and reduced WGIUH obtained by considering only the coloured points in Figure 5 which are also connected with the outlet by a path made of coloured pixels. As a consequence, a different maximum discharge will correspond to every degree of saturation.

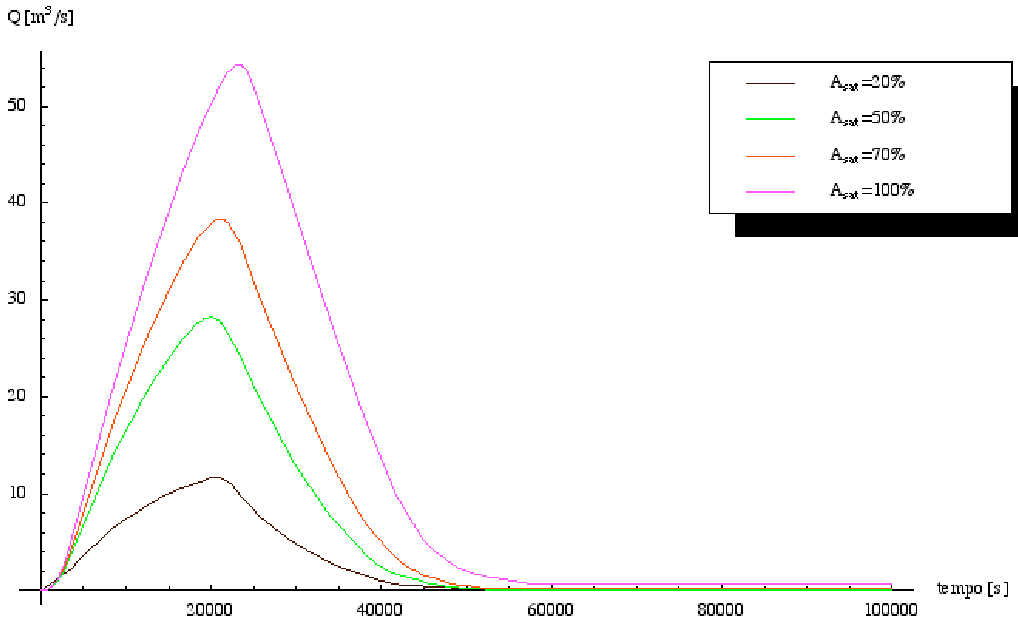

Figure 6 – Shows the liquid discharge for a velocity ratio of 20 (c=2 m/s and c_h=0.1 m/s) for different portions of saturated areas of the basin.

The very maximum flood will occur (kept the effective rainfall fixed) obviously with the maximum of saturated areas. In the limit of having all the basin saturated, the discharge obtained by the Dunnian mechanism will coincide with the one obtained by Hortonian runoff, in spite of the different dynamical mechanism acting. A reasonable Dunnian guess is that in extreme events the whole concave area is saturated and this reduces the maximum real discharge to less than 40% of the Hortonian maximum discharge.

Conclusion

It is shown here how to construct rescaled width functions and the corresponding WGIUH with the help of the Grass GIS and some custom codes. For the first time the concept of a variable area width function depending from on the saturation degree of the basin has been introduced. It is also briefly shown how from the WGIUH it is obtained the discharge. However users must be warned that to obtain the state of art reproduction of flood waves, it is necessary to introduce some further dynamic parameters in order to account for hydrodynamical dispersion (Rinaldo et al, 1995). The custom code, which is available upon request to the authors, is being ported to GRASS soon.

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